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Orderings of one-dimensional Ising systems with an arbitrary interaction of finite range

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Abstract. The concept of the *irreducible block* is introduced for the one-dimensional Ising system with an arbitrary interaction of finite range. It is proved that the ground state energy of the system occurs for a regular chain of one of the irreducible blocks or for coexistence of the regular chains of the irreducible blocks. The theorems given in the text involve statements as to when one has a regular ordering and when irregular orderings appear. Some of our conclusions are (i) that the ground state energy of the linear Ising magnet with pair interactions up to third neighbours is effected by the seven spin orderings recently given for the case of spin $\frac{1}{2}$ by Katsura and Narita, and (ii) that the ground state energy of the Ising magnet of spin larger than $\frac{1}{2}$ is effected by the same set of orderings as for the corresponding Ising magnet of spin $\frac{1}{2}$.

1. Introduction

It is well known that the Ising magnet of spin $\frac{1}{2}$ orders either ferromagnetically or antiferromagnetically if the nearest neighbour interaction is assumed. If the interaction is up to second neighbours and a uniform external field is applied, four orderings occur (Oguchi 1965). Morita and Horiguchi (1972) gave an elementary method for proving this fact.

In general we believe that a system takes a simple regular structure in the ground state, and tries to get the ground state as the state which has the lowest energy of the various simple regular orderings. Such an attempt has been presented by a number of people for various systems with an interaction of a range longer than the nearest neighbours; eg, Luttinger and Tisza (1946), Meijer and Niemeijer (1973) for classical spins coupled by the dipolar force; see Nagamiya (1967) for classical spins coupled by the Heisenberg interaction; Horiguchi and Morita (1972), Katsura and Narita (1973a) for the square and SC Ising magnet of spin $\frac{1}{2}$. In another recent paper, Katsura and Narita (1973b) investigated the one-dimensional Ising magnet of spin $\frac{1}{2}$ when the interaction is up to the third neighbours; they assumed that the system orders with a repeating unit composed of six or less spins and obtained seven orderings.

Luttinger and Tisza (1946) conjectured that the ground state has the translational symmetry by the vectors which are equal to twice the lattice constants, for the classical spins interacting via the dipolar force. Recently, Karl (1973) gave a proof justifying this conjecture for the systems in which the interaction is the nearest neighbour one and some symmetry properties are satisfied.

In the present paper, we shall focus our attention on one-dimensional Ising systems with an arbitrary interaction of finite range. We extend the above-mentioned argument of Morita and Horiguchi (1972) and present a number of theorems in which it is stated that the ground state of this system is effected by a regular chain of a repeating unit of finite length or by coexistence of such regular chains. The theorems involve statements as to when one has a regular ordering and when irregular orderings appear. By applying the theorems, we prove that the Ising magnet of spin $\frac{1}{2}$ with an interaction up to third neighbours orders in the seven ways which were given by Katsura and Narita (1973b).

The discussions in the present paper are mostly for the one-dimensional Ising systems with an interaction of finite range. Section 6 is an exception where we consider the Ising magnet of spin larger than $\frac{1}{2}$, in an arbitary lattice and with an interaction of an arbitrary range. Under the assumption that the energy of the system is linear in each spin, we prove that the ground state energy of the Ising magnet of spin S, which is larger than $\frac{1}{2}$, can be effected by an ordering in which no lattice sites take the configuration other than spin $-S$ and S. As a consequence, we conclude that this system takes the same sets of orderings as for the corresponding Ising magnet of spin $\frac{1}{2}$. In fact, for the linear Ising magnet of spin 1 with an interaction up to second neighbours, Katsura and Narita (1973b) obtained the same four orderings as for the case of spin $\frac{1}{2}$.

2. Energy of a chain in an arbitrary arrangement

We consider a general one-dimensional lattice system in which each lattice site takes on a finite number s_M of configurations. We assume an interaction of finite range r. If the total number of lattice sites is L, the total energy of the system in an arbitrary arrangement is expressed as follows:

$$E = \sum_{i=1}^{L-r} \phi^{(r+1)}(s_i, s_{i+1}, \ldots, s_{i+r}) + \phi_L(s_{L-r+1}, s_{L-r+2}, \ldots, s_L) \tag{1}$$

where s_i denotes the configuration on the ith lattice site, and $\phi^{(r+1)}$ takes the following form:

$$\begin{aligned}\phi^{(r+1)}&(s_i, s_{i+1}, \ldots, s_{i+r}) \\ &= u^{(1)}(s_i) + \sum_{i<j\leqslant i+r-1} u^{(2)}_{j-i}(s_i, s_j) + \sum\sum_{i<j<k\leqslant i+r-1} u^{(3)}_{j-i,k-j}(s_i, s_j, s_k) \\ &\quad + \ldots + u^{(r+1)}_{1,1,\ldots,1}(s_i, s_{i+1}, \ldots, s_{i+r}).\end{aligned}$$

ϕ_L is an arbitrary correction to the total energy due to the last r lattice sites.

We shall pay attention to a block of sites, eg, of l sites from i to $i+l-1$. If the configurations of the r sites starting from i and from $i+l$ are isomorphic, that is, if

$$s_{i+j} = s_{i+l+j}, \qquad 0 \leqslant j \leqslant r-1, \tag{2}$$

then we call the block of the l sites from i to $i+l-1$ a *displaceable block*. If a chain of sites involves a displaceable block, we call the chain *reducible*. If the chain is not reducible, we call it *irreducible*. As to the existence of displaceable blocks, we note the following lemma.

Lemma 1. A chain of $s_M^r + r$ lattice sites always involves a displaceable block composed of s_M^r or less sites and is reducible.

This fact is seen as follows: the total number of all the possible configurations for a segment of r sites is s_M^r, and hence if we consider s_M^r+1 segments of r sites starting from 1st, 2nd, ..., and (s_M^r+1)th site in the chain of s_M^r+r sites, then at least two of the segments must take the same configuration; that means an existence of a displaceable block composed of s_M^r or less sites in the chain.

From a block of l sites, we construct a *regular chain* of the block in which every segment of l sites starting from site $ml+1$ ($m = 0, 1, 2, \ldots$) takes the same configuration as the block under consideration.

We shall consider deleting a displaceable block from a chain. After deleting the block, we shall regard the site just following the block as the next site to the site just preceding the block (if it exists) in the remaining chain. We can confirm the following lemma.

Lemma 2. If E' denotes the energy of the chain obtained after deleting a displaceable block D from a chain, then the energy of the original chain E is expressed as follows:

$$E = \epsilon_{\mathrm{D}}+E'$$

where ϵ_{D} is the energy per block in the infinite regular chain of the block D.

We shall associate the energy $\phi^{(r+1)}(s_i, s_{i+1}, \ldots, s_{i+r})$ to the ith lattice site. If the total number of sites l involved in the deleted displaceable block D is equal to or more than r, we see that lemma 2 is trivial. If l is less than r, we use (2) for $j = l+j'$ and show that

$$s_{i+j'} = s_{i+l+j'} = s_{i+2l+j'}, \qquad 0 \leqslant j' \leqslant r-l-1.$$

In general, we find that

$$s_{i+j} = s_{i+kl+j} \qquad \begin{cases} 1 \leqslant k \leqslant [r/l], & 0 \leqslant j \leqslant l-1 \\ k = [r/l]+1, & 0 \leqslant j \leqslant r-[r/l]l-1, \end{cases} \tag{3}$$

where $[r/l]$ is the integral part of r/l. This shows that the arrangement in the segment of r sites following the deleted displaceable block D in the chain is the same as in the corresponding segment in the regular chain of the block D. With this consideration, we conclude lemma 2 holds also for the case of $l < r$.

The following lemma is trivial.

Lemma 3. If N_s' is the total number of lattice sites taking the configuration s in the chain which is obtained from a chain by deleting a displaceable block D, then the total number of sites taking the configuration s in the original chain is expressed as follows:

$$N_s = \nu_{\mathrm{D}s}+N_s'$$

where $\nu_{\mathrm{D}s}$ is the total number of sites taking the configuration s in the block D.

If the regular chain of a block A involves a displaceable block B of a smaller number of sites than A, we call the original block A *reducible*. If the block is not reducible, we call it *irreducible*. From this definition and lemma 1, the following lemma follows.

Lemma 4. An irreducible block is an irreducible chain and the total number of the lattice sites involved in it is less than or equal to s_M^r.

We note the following lemma.

Lemma 5. Every displaceable reducible block in a chain involves a displaceable irreducible block of the same chain.

In order to show this lemma, we consider a displaceable reducible block A constituted of l_A sites $i, i+1, \ldots$, and $i+l_A-1$. By assumption, we have a displaceable block B of sites $i+j, i+j+1, \ldots, i+j+l_B-1$ in the regular chain of the block A, where $l_A > j \geqslant 0$ and $l_B < l_A$. In that case, we note that l'_B lattice sites $i+j+l_B-l_A, i+j+l_B-l_A+1, \ldots$, and $i+j-1$ also constitute a displaceable block B′ in the same regular chain, where $l'_B = l_A - l_B$. According as $j+l_B \leqslant l_A$ or $j+l_B > l_A$, the block B or B′ is a displaceable block involved in the block A of the original chain. With this consideration, we conclude the existence of a displaceable irreducible block in the block A, if we recall the facts that a block is either reducible or irreducible and a block of one site is always irreducible.

We consider an arbitrary chain. Starting from the chain, we delete displaceable irreducible blocks. By virtue of lemma 5, we see that after this process, we finally reach an irreducible chain, which cannot be longer than $s_M^r + r - 1$ by lemma 1. With the aid of the lemmas 2 and 3, we now obtain the following basic theorem.

Theorem 1. The total energy of a chain can always be expressed as follows:

$$E = \sum_{B} n_B \epsilon_B + E'' \tag{4}$$

where the summation on the right hand side is taken over all the irreducible blocks B. ϵ_B is the energy per block in the infinite regular chain of the block B and E'' the energy for an irreducible chain C_F. n_B are zero or positive integers. If ν_{Bs} denotes the number of lattice sites taking configuration s in the block B, then the total number of the lattice sites taking the configuration s in the chain is expressed as follows:

$$N_s = \sum_{B} n_B \nu_{Bs} + N''_s \tag{5}$$

where N''_s is the number of sites taking the configuration s in the chain C_F.

Taking the summation over s of (5), the total number of lattice sites L in the chain is expressed as

$$L = \sum_{B} n_B \nu_B + N'' \tag{6}$$

where ν_B and N'' are the total numbers of sites involved in the block B and in the chain C_F, respectively.

3. Ground state of a very long chain

There are two types of problems in determining the ground state of a system. Case I when only the total number of the lattice sites L in the chain is given. Case II when not only the L but also the total numbers of lattice sites N_s in each configuration s in the chain are given.

We first consider case I. According to theorem 1 of the preceding section, the energy of a chain of L lattice sites cannot become smaller than the minimum value of the expression (4) obtained under the restriction (6). We assume that L is very large. Let $(\epsilon_B/\nu_B)_{\min}$ be the smallest value of the energy per lattice site ϵ_B/ν_B of all the irreducible blocks. If the minimum value $(\epsilon_B/\nu_B)_{\min}$ occurs for only one B, then $n_B\nu_B$ is put equal to $L-N''$ for that block B and to zero for all the other blocks. If $(\epsilon_B/\nu_B)_{\min}$ occurs for two or more B, then the lowest energy of (4) is effected by putting the sum of $n_B\nu_B$ for these B equal to $L-N''$ and $n_B\nu_B$ for all the other B to zero. In fact, we conclude the following theorems.

Theorem 2. If the minimum value of ϵ_B/ν_B occurs for only one of all the irreducible blocks, the ground state of a very long chain corresponds to the regular chain of that one irreducible block B.

Theorem 3. If the minimum value of ϵ_B/ν_B occurs for σ ($\sigma \geqslant 2$) of the irreducible blocks and if no pair of all the σ regular chains of each of those σ blocks has no isomorphic segment composed of r sites, then the ground state corresponds to an arbitrary coexistence of the σ regular chains, where r is the range of the interaction. If there occurs an isomorphic segment composed of r sites in a pair of the σ regular chains, then an irregular chain may appear within a part of the chain in the ground state.

This conclusion is reached by observing that, if an isomorphic segment composed of r sites occurs in a pair of the regular chains of B_1 and B_2 we can make a mixed chain composed of irreducible blocks B_1 and B_2, without changing the energy per site of the chain.

In case II, the minimum of (4) is taken under the restriction of (5). When N_s/L are given, s_M equations (5) can be satisfied if we assume that all n_B excluding s_M of them are zero. For every set of s_M B, we assume that all n_B except for those B are zero and calculate the s_M values of n_B. If all these s_M values of n_B are determined to be non-negative, we calculate E/L by (4). We shall denote the minimum of the thus-calculated values of E/L by $(E/L)_{\min}$. If σ ($\sigma \leqslant s_M$) of n_B are found positive among the set of s_M n_B giving the value $(E/L)_{\min}$ in the above calculation, we shall denote the set of the σ B for which n_B are found positive, by $(B_1, B_2, \ldots, B_\sigma)_{\min}$. We can then confirm that E/L given by (4) cannot take a smaller value than $(E/L)_{\min}$ and conclude the following theorem.

Theorem 4. In the case when the total numbers of lattice sites N_s in each of s_M configurations are prescribed, if all the sets of $(B_1, B_2, \ldots, B_\sigma)_{\min}$ corresponding to the given set of N_s are given and if no pair of the regular chains of each of $B_1, B_2, \ldots,$ and B_σ have no isomorphic segment composed of r sites, then the ground state is given by the state of coexistence of σ regular chains of the irreducible blocks $B_1, B_2, \ldots,$ and B_σ, respectively. If there exists only one set $(B_1, B_2, \ldots, B_\sigma)_{\min}$, then the length of each regular chain is uniquely determined. If a pair of the regular chains of each of $B_1, B_2, \ldots,$ and B_σ has an isomorphic segment composed of r sites, an irregular chain may appear within a part of the chain.

Theorems 2–4 state that the orderings in the ground state of the one-dimensional Ising systems can be determined from knowledge about the irreducible blocks.

All the irreducible blocks $[s_1 s_2 \ldots s_l]$ are given below for the cases of $s_M = 2$ and $1 \leqslant r \leqslant 3$. If $r = 1$, the set is [1], [2], [12]. If $r = 2$, we further have [121], [122], [1221]. If $r = 3$, we further have [1211], [12211], [1222], [12221], [122211], [121221], [1212211], [1212221], [12122211]. If $r = 4$, we have further 91 irreducible blocks.

When $s_M = 3$, we have all the blocks given for the case of $s_M = 2$ and [3] and all those which are obtained from the ones involving both 1 and 2 by replacing 1 and 2 either by 2 and 3, respectively, or by 1 and 3, respectively. If $r = 1$, we have [1], [2], [3], [12], [13], [23], and [123]. If $r = 2$, we have further 80 irreducible blocks. The longest irreducible blocks are found to have s_M^r lattice sites for all the cases when the listing of them is performed; for $s_M = 2$, $1 \leqslant r \leqslant 3$ and $s_M = 3$, $1 \leqslant r \leqslant 2$.

In the procedure of producing all the irreducible blocks by a computer, we first obtain all the irreducible chains and then delete those which are not irreducible blocks from them. The procedure is devised to retain only one from all the equivalent blocks which are achieved from others by a rotation and/or a reflection.

4. Ground state of the Ising magnet

We now consider the Ising magnet of spin S and with the pair interaction of a finite range r. The energy of this system is given by

$$E = -h \sum_{i=1}^{L} s_i - \sum_{\substack{1 \leqslant i < j \leqslant L \\ j-i \leqslant r}} J_{j-i} s_i s_j \tag{7}$$

where s_i takes on the values $-S$, $-S+1, \ldots,$ and S. We assume that h is positive without loss of generality.

In determining the ground state of the Ising magnet, the total numbers of lattice sites occupied by each value of spin from $-S$ to S are not given. The only restriction is (6), and theorems 2 and 3 of the preceding section apply. The ground state ordering is determined by the irreducible blocks B for which the energy per lattice site ϵ_B/ν_B is minimum among all the irreducible blocks. If the minimum of ϵ_B/ν_B occurs for only one B, the ground state corresponds to the regular chain of that block B.

We notice that, if we change the signs of all the spins involved in an irreducible block, we obtain again an irreducible block. Considering this fact, we note that the block with more negative spins than positive ones cannot be the block with minimum ϵ_B/ν_B, if $h > 0$, and we shall exclude those. From the list for the case of $s_M = 2$ in the preceding section, we then have the following set of the irreducible blocks for the case of $S = \frac{1}{2}$: $(+)$, $(+-)$ if $r = 1$ and further $(+-+)$, $(+--+)$ if $r = 2$. For $r = 1$ and 2, we know that the regular chains of these two and four irreducible blocks become the ground state in some regions on the J_1/h line and in the $(J_1/h, J_2/h)$ plane, respectively (Oguchi 1965, Morita and Horiguchi 1972).

If $r = 3$, we have further $(+-++)$, $(+--++)$, $(+---++)$, $(+-+--+)$, $(+-+--++)$, $(+-+----++)$. We find that there exists no set of J_1, J_2 and J_3 when the last three take the minimum value of ϵ_B/ν_B; eg an inconsistency results if ϵ_B/ν_B is assumed to be smaller (i) for $(+-+--+)$ than for each of $(+-)$, $(+-+)$ and $(+--+)$, (ii) for $(+-+--++)$ than for each of $(+-)$, $(+--+)$ and $(+-++)$, and (iii) for $(+-+----++)$ than for each of $(+-)$ and $(+---++)$. When the external field h is positive and the interaction is up to third neighbours, the present calculation thus proves that all the possible orderings of the Ising magnet of spin $\frac{1}{2}$ are the seven, excluding the last three among the ten listed above. In fact, Katsura and Narita (1973b) compared the energies of the regular orderings of blocks composed of six or less lattice sites, and gave the regions where all those seven orderings become the ground state in the $(J_1/h, J_2/h, J_3/J_1)$ space.

On the boundaries between two, three and four regions in the J_1/h line, $(J_1/h, J_2/h)$ plane, and $(J_1/h, J_2/h, J_3/h)$ space, which are for the cases $r = 1, 2, 3$, respectively, the minimum value of ϵ_B/ν_B is taken by two, three and four B, respectively. Theorem 3 applies for those cases. Then we conclude the occurrence of coexistent states of the regular chains. Sometimes irregular chains are also expected.

5. Ground state of binary mixtures

In this section, we consider binary mixtures on a lattice. Each lattice site is occupied by one particle. We assume a pair interaction of range r between particles. When the total numbers of particles N_s of each species s ($s = 1$ or $s = 2$) are given, the total energy of an arbitrary arrangement of the system can be expressed in the following form (eg see ter Haar 1954):

$$E_{\text{tot}} = E^{(0)} + \sum_{s=1}^{2} N_s E_s^{(1)} + E \tag{8}$$

where

$$E = \sum_{\substack{1 \leqslant i < j \leqslant L \\ j-i \leqslant r}} \sum J_{j-i} s_i' s_j'. \tag{9}$$

Here s_i' takes the values $+1$ or -1 according as the site i is occupied by a particle of species $s = 1$ or $s = 2$. $E^{(0)}$, $E_s^{(1)}$, J_{j-i} are constants determined by the interaction.

In this problem, $N_1 \equiv N_+$ and $N_2 \equiv N_-$ are given and theorem 4 of § 3 applies. As seen in § 3, the total number of the possible orderings is equal to the total number of all the different irreducible blocks, and that number is three and six if the range of interaction r is one and two, respectively.

The graph of E/L against N_-/L is given for the case $r = 2$ in figure 1. In the graph, we plot the points $(\nu_{B-}/\nu_B, \epsilon_B/\nu_B)$ for all B, and then draw straight lines connecting all the pairs of these points. When the line, connecting the points for the irreducible blocks B_1 and B_2, gives the lowest value of E/L for the given value of N_-/L, the point on the line gives the lowest value of E/L for that N_-/L. The ground state for an arbitrary energy is effected by the coexistence of the two regular chains of each of the B_1 and B_2. Figure 1 gives this graph for various values of J_2 and J_1 as indicated in figure 2. Figure 2 shows the change of the orderings as a function of N_-/L for an arbitrary J_2/J_1. If J_1 and J_2 are given, we see the radial direction in figure 2 corresponding to the ratio J_2/J_1. If the given value of N_-/L drops between two of the arcs and/or circles, the system takes the coexistent state of the two orderings corresponding to those two arcs and/or circles. If $J_2 > 0$, there exists no isomorphic segment composed of two sites in the regular chains of $+$ and $-$, respectively, or in the regular chains of $+\,-$ and either of $+$ or $-$, and hence the ground state is a coexistent state of those two regular chains, for an arbitrary value of N_-/L. For the other values of $J_2 < 0$, there exists an isomorphic segment composed of two sites in those two chains if N_-/L is between two arcs and/or circles, and hence an irregular chain may occur in the chain in the ground state. If N_-/L is on one of the arcs and circles, one has the regular chain of the block corresponding to that arc or circle. When J_1 or J_2 is zero, there will occur further complex chains. We note that the part of $N_-/L < \frac{1}{2}$ in the present figure 2 corresponds to figure 1 in our preceding note (Morita and Horiguchi 1972).

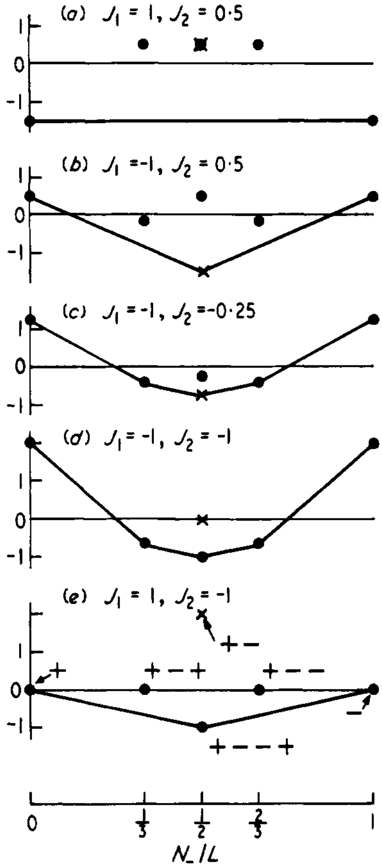

Figure 1. E/L against N_-/L for the binary mixtures of species + and − with the pair interaction up to second neighbours. The black circles on the line of $N_-/L = 0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, and 1 give ϵ_B/ν_B for the irreducible blocks B = +, + − +, + − − +, + − −, and −, respectively. The crosses on $N_-/L = \frac{1}{2}$ are for B = + −.

For the case of an interaction up to third neighbours, Katsura and Narita (1973b) gave the figures corresponding to figure 1 of our preceding note. One can easily construct the figures corresponding to the present figure 2 for that case from their figures. Figure 3 gives an example.

6. Ising magnet of spin larger than $\frac{1}{2}$

In the present section, we consider the Ising magnet of spin larger than $\frac{1}{2}$. We shall not restrict ourselves to the linear chain nor to the interaction of finite range in the main discussion in this section. The following proposition is easily proved.

Theorem 5. If the energy of the system is linear in each of the spins, then the ground state energy of the Ising magnet of spin S which is larger than $\frac{1}{2}$, is effected by an ordering which consists only of spin S and $-S$. In particular, for the case of the linear Ising magnet, the ground state is effected by the regular chain of one of the irreducible blocks which consist only of spin S and $-S$.

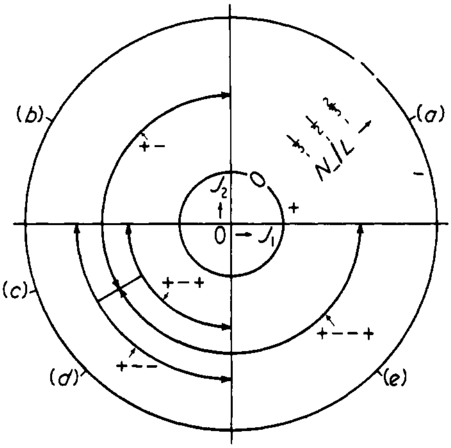

Figure 2. Phase diagram showing the orderings of the binary mixtures of species + and − with the pair interaction up to second neighbours, as a function of N_-/L, J_1 and J_2. (*a*)–(*e*) on the periphery show the ratios of J_2 and J_1 for which the energy E/L is given as a function of N_-/L in figure 1.

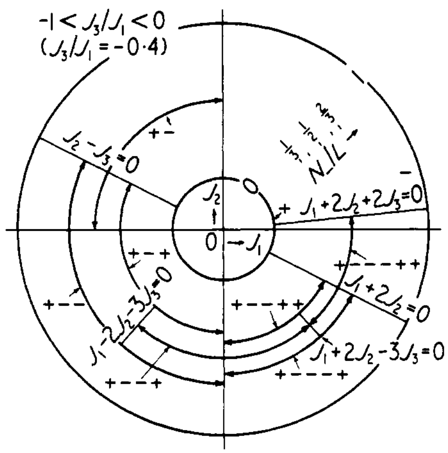

Figure 3. Phase diagram showing the orderings of the binary mixtures of species + and − with the pair interaction up to third neighbours. This figure is for $J_3/J_1 = -0{\cdot}4$. The phase diagram takes the same form topologically in the range $-1 < J_3/J_1 < 0$.

Let us assume that the spin s_i at the ith site is not equal to $-S$ nor to S in an ordering C_0. We shall consider the orderings C_+ and C_- which are obtained from the original one C_0 by replacing the spin s_i on the ith site by spin S and $-S$, respectively. We note that the energy of the ordering C_0 cannot be smaller than both of C_+ and C_-. This fact implies that the ground state energy of the Ising magnet of spin larger than $\frac{1}{2}$ can be effected by the spin orderings composed only of spin $+S$ and $-S$. The problem of obtaining the ground state orderings of spin $+S$ and $-S$ is thus reduced to the problem of the Ising magnet of spin $\frac{1}{2}$.

The only possible situation when a spin not equal to $-S$ nor to S may appear in the ground state is the case when the energy does not change if we change the spin at a site from $+S$ to $-S$ or *vice versa* in an ordering with the ground state energy. In that case we may replace the spin at that site by an arbitrary spin without changing the energy. For the one-dimensional case with an interaction of finite range, such may happen at the boundaries of the regions in the $(J_1/h, J_2/h, \ldots, J_r/h)$ space. For instance, if we assume $J_1 = -h/2$ and $J_2 > 0$ for the linear Ising magnet of spin 1 with an interaction up to second neighbours, an irregular chain involving spin 0 may occur in the ground state.

It is now trivial to say that the same two, four and seven orderings for the linear Ising magnet of an arbitrary spin occur as for the Ising magnet of spin $\frac{1}{2}$, if the interaction is up to first, second and third neighbours, respectively.

7. Ternary mixtures

For one-dimensional ternary mixtures, theorem 4 applies. The following description gives a process of obtaining the set $(\mathrm{B}_1, \mathrm{B}_2, \ldots, \mathrm{B}_\sigma)_{\min}$. We draw a graph of E/L on the $(N_1/L,\ N_2/L,\ N_3/L)$ plane. At the point $(\nu_{\mathrm{B}1}/\nu_{\mathrm{B}}, \nu_{\mathrm{B}2}/\nu_{\mathrm{B}}, \nu_{\mathrm{B}3}/\nu_{\mathrm{B}})$, we take the height $\epsilon_{\mathrm{B}}/\nu_{\mathrm{B}}$; see figure 4. For each set of three B, we draw the triangular plane which has the vertices at the points $(\nu_{\mathrm{B}1}/\nu_{\mathrm{B}}, \nu_{\mathrm{B}2}/\nu_{\mathrm{B}}, \nu_{\mathrm{B}3}/\nu_{\mathrm{B}}, \epsilon_{\mathrm{B}}/\nu_{\mathrm{B}})$ for those B. For a given set of values of N_1/L, N_2/L and N_3/L, the minimum of $(E/L)_{\min}$ of the values of E/L on those triangular planes gives the ground state energy. If the vertices of the triangular plane giving the lowest value correspond to the irreducible blocks B_1, B_2, and B_3, the state is represented by the coexistence of the three regular chains of these blocks $\mathrm{B}_1, \mathrm{B}_2$, and B_3.

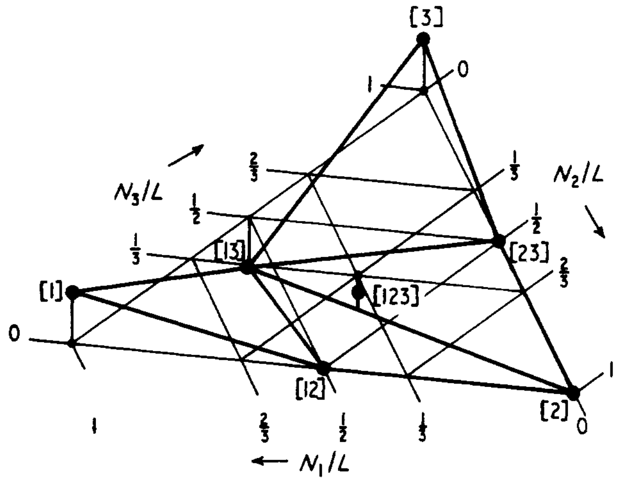

Figure 4. Graph of E/L on the $(N_1/L,\ N_2/L,\ N_3/L)$ plane for a ternary mixture with the nearest neighbour interaction. The values of $\epsilon_{\mathrm{B}}/\nu_{\mathrm{B}}$ at $(\nu_{\mathrm{B}1}/\nu_{\mathrm{B}}, \nu_{\mathrm{B}2}/\nu_{\mathrm{B}}, \nu_{\mathrm{B}3}/\nu_{\mathrm{B}})$ are shown by black circles for B = [1], [12], [13], [23], and [123], respectively.

As an illustration, we consider a ternary mixture with a nearest neighbour interaction. The total energy E_{tot} of the chain in an arbitrary arrangement is given by

$$E_{\mathrm{tot}} = E_0 + \sum_{s=1}^{3} N_s u^{(1)}(s) + E$$

where

$$E = \sum_{i=1}^{L-1} u^{(2)}(s_i, s_{i+1}). \tag{10}$$

We shall assume that

$$u^{(2)}(1,1) = u^{(2)}(3,3) = 1,$$

$$u^{(2)}(1,3) = u^{(2)}(3,1) = -1,$$

$$u^{(2)}(2,s) = u^{(2)}(s,2) = 0, \qquad s = 1,2,3.$$

The graph of E/L on the plane of $(N_1/L, N_2/L, N_3/L)$ is given in figure 4. This figure shows that the ground state of this system is given by the coexistence of the regular chains (i) of [1], [12], and [13] if $N_1/L > \frac{1}{2}$, (ii) of [3], [13], and [23] if $N_3/L > \frac{1}{2}$, (iii) of [2], [12], and [13] if $\frac{1}{2} > N_1/L > N_3/L$, and (iv) of [2], [13], and [23] if $\frac{1}{2} > N_3/L > N_1/L$.

We shall consider the case when all $u^{(2)}(s, s')$ in (10) for $s, s' = 1, 2, 3$ have the opposite signs. Figure 4 should then be made upside down. The triangular plane giving the lowest energy is the one which has vertices at the positions of the blocks [1], [2] and [3] for all values of $(N_1/L, N_2/L, N_3/L)$, and hence the ground state is always represented by the coexistence of the regular chains of the blocks [1], [2], and [3], respectively.

8. Summary and comments

The concept of irreducible blocks is introduced. It is proved that the ground state energy of a one-dimensional Ising magnet is effected by a regular chain of the irreducible block. For the case of one-dimensional lattice gas mixtures, the ground state is effected by a coexistent state of the regular chains of the irreducible blocks. The theorems presented in § 3 involve the criterion on the irreducible blocks, as to when we have a regular chain, when we have a coexistent state of a number of regular chains, and when an irregular chain occurs in the ground state.

Those theorems are applied to the Ising magnet of spin $\frac{1}{2}$ and of an arbitrary spin in §§ 4 and 6, respectively, and to the binary and ternary mixtures in §§ 5 and 7, respectively.

Before closing this paper, we give some comments on the square and SC Ising magnets. The proof of Luttinger and Tisza's conjecture due to Karl (1973) applies to the square lattice with an interaction up to second neighbours and to the SC lattice with an interaction up to third neighbours. Thus we confirm that the spin orderings given for the Ising magnet of spin $\frac{1}{2}$ by Horiguchi and Morita (1972) and by Katsura and Narita (1973a) are all the orderings occurring in these systems. If we recall theorem 5 given in § 6, we notice that those orderings are also all for the Ising magnet of spin larger than $\frac{1}{2}$.

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